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THE UNIQUENESS AND STABILITY OF THE SOLUTION OF THE RIEMANN PROBLEM OF A SYSTEM OF CONSERVATION LAWS OF MIXED TYPE

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ABSTRACT. We establish the uniqueness and stability of the similarity solution of the Riemann problem for a 2×2 system of conservation laws of mixed type, with initial data separated by the elliptic region, which satisfies the viscosity-capillarity travelling wave admissibility criterion.

1. INTRODUCTION

The isothermal evolution of one-dimensional continuous compressible media in the absence of body forces can be described in Lagrangian coordinates by the quasilinear system of conservation laws

$$u_t + p(w)_x = 0, \tag{1.1a}$$

$$w_t - u_x = 0, \qquad x \in \mathbb{R}, \ t > 0, \tag{1.1b}$$

where $p(w)$ is the pressure. Typically, for instance in ideal gases, $p'(w) < 0$, so that the system (1.1a, b) is hyperbolic. For some other material models, for example the van der Waals gas or elastic/plastic rods, $p'(w)$ may be positive on some range of w, as depicted in Figure 1. More precisely, we assume

$$\begin{aligned} p(w) \in C^1(\mathbb{R}) \quad \text{and} \quad p'(w) < 0 \quad \text{for } w \notin [\alpha, \beta], \\ p'(w) > 0 \quad \text{for } w \in (\alpha, \beta). \end{aligned} \tag{1.1c}$$

With this kind of function $p(w)$, the system (1.1a, b) is of hyperbolic-elliptic mixed type.

In this paper, we shall continue the program carried out in [20, 7] to study the system (1.1a, b, c) with the following Riemann initial values

$$(u(x, 0), w(x, 0)) = \begin{cases} (u_-, w_-) & \text{for } x < 0, \\ (u_+, w_+) & \text{for } x > 0, \end{cases} \tag{1.1d}$$

$$w_- < \alpha < \beta < w_+. \tag{1.1e}$$

The system (1.1) generally admits many solutions but not every one of them is physically relevant. This raises the issue of the admissibility of these solutions.

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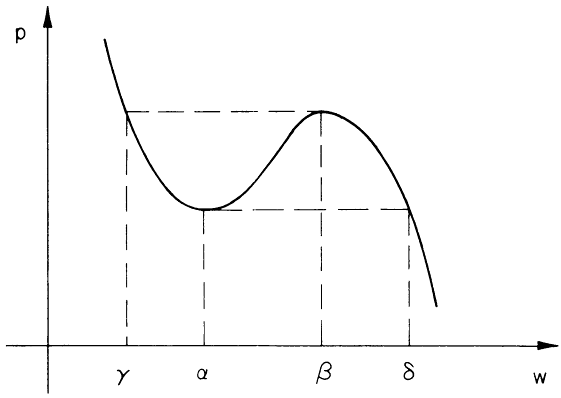

FIGURE 1

In other words, we have to develop some admissibility criterion to single out the "physically correct" solution of (1.1), or better yet, to establish the well-posedness for the Cauchy problem for (1.1a, b).

In the context of hyperbolic systems, many admissibility criteria have been proposed. An early example is the Lax shock admissibility criterion [13]. A comprehensive shock admissibility criterion was proposed by Liu [14] which yields a satisfactory solution of the Riemann problem for strictly hyperbolic systems when the waves are of moderate strength. Based on the premise that admissibility should be invariant under translations and dilatations, Dafermos presented, in his recent paper [5], the wave fan admissibility criterion. Dafermos also proposed the entropy rate criterion in [1], and proved in [5] that, for wave fans of moderate strength, the entropy rate criterion and the Liu admissibility condition are equivalent.

A successful criterion for mixed type systems should not only comply with the established criteria for hyperbolic systems for the part of solutions of (1.1) inside a connected component of the hyperbolic region, but should also satisfy physical principles governing phase transitions, for example the Maxwell equal area rule, as well as agree with experimental results. Slemrod [18] suggested, in the context of (1.1a, b), the viscosity-capillarity travelling wave criterion, or travelling wave criterion for short, which meets the above standards quite nicely [8, 19, 22]. Based on Korteweg's theory, Slemrod's criterion states that a shock of (1.1), (u_1, w_1), (u_2, w_2), satisfying, of course, the Rankine-Hugoniot condition, is admissible if

$$(1.2)\quad \begin{gathered} A\frac{d^2\hat{w}}{d\zeta^2} = -s\frac{d\hat{w}(\zeta)}{d\zeta} - D\left(\frac{d\hat{w}(\zeta)}{d\zeta}\right)^2 - p(\hat{w}) + p(w_1) - s^2(\hat{w}(\zeta) - w_1), \\ \hat{w}(-\infty) = w_1, \quad \hat{w}(+\infty) = w_2, \quad \hat{w}'(\pm\infty) = 0 \end{gathered}$$

has a solution, where A, D are constants and s is the speed of the shock. A solution of (1.1) is admissible according to the travelling wave criterion if each discontinuity of the solution is a jump discontinuity and is admissible by the travelling wave criterion. For a recent survey of the travelling wave theory of the dynamics of phase transitions, the reader is referred to [22].

The common approach for solving the Riemann problem (1.1) is to construct the admissible shock and wave curves. If it succeeds, this approach gives us centered wave solutions. Discussions using this procedure have been given by R. James [11] and later by M. Shearer [15–17]. L. Hsiao [9, 10] also studied this problem using other admissibility criteria. At this stage, it is natural to ask questions about the existence, uniqueness, and stability of these solutions.

The existence of solutions of (1.1) which are admissible according to the travelling wave criterion (1.2) when $A = 1/4$ and $D = 0$ was established recently by Fan in [7]. Following Slemrod [20], Fan [7] constructed solutions of (1.1) as the $\varepsilon \to 0+$ limit of solutions of the system

$$u_t + p(w)_x = \varepsilon t u_{xx}, \tag{1.3a}$$

$$w_t - u_x = \varepsilon t w_{xx}, \tag{1.3b}$$

$$(u(x, 0), w(x, 0)) = \begin{cases} (u_-, w_-) & \text{for } x < 0, \\ (u_+, w_+) & \text{for } x > 0. \end{cases} \tag{1.3c}$$

The above "similarity viscosity" approach was pursued by Kalasnikov [12], Tupciev [23, 24], Dafermos [2, 3, 5], Dafermos and DiPerna [4], Slemrod [20], Slemrod and Tzavaras [21], and Fan [6, 7]. For convenience, we shall, by saying that a solution of (1.1) is admissible according to the *similarity viscosity admissibility criterion*, mean that this solution is constructed by the above similarity viscosity approach. In [7], the author proved the existence of centered wave solutions $(u(x/t), w(x/t))$ of (1.1) which possess one phase change and satisfying the similarity viscosity admissibility criterion under the assumption that

$$|p(w)| \to \infty.$$

These solutions are also admissible according to the travelling wave criterion with $A = 1/4$ and $D = 0$ if each straight line in the (w, p) plane intersects the graph of $p(w)$ at at most finite points.

In this paper, we assume the following

Assumption 1. Besides (1.1c), $p(w)$ also satisfies

$$p''(w) > 0 \quad \text{for } w \le \alpha, \tag{1.4a}$$

$$p''(w) < 0 \quad \text{for } w \ge \beta. \tag{1.4b}$$

We shall see in §2 that under Assumption 1 solutions to the Riemann problem (1.1), satisfying the travelling wave criterion, possess one and only one phase boundary. In the sequel of this paper, when we say solutions of (1.1) we always refer to centered wave solutions of (1.1) which are admissible according to the travelling wave criterion with $A = 1/4$, $D = 0$.

Under Assumption 1, we shall prove the uniqueness and stability of the solution of (1.1). In §2, we study the structure of the solution. We devote §3 to the study of phase boundaries. After this preparation, we shall prove, in §4, our main result:

Theorem 1.1. (i) (1.1) *has a unique solution within the class of centered wave solutions satisfying the travelling wave criterion.*

(ii) *Let* $(u(\xi), w(\xi))$ *be the solution of* (1.1). *For any* $\varepsilon > 0$ *and* $\gamma > 0$, *there is a* $\delta > 0$ *such that if*

$$|u_- - \bar{u}_-| + |u_+ - \bar{u}_+| + |w_- - \overline{w}_-| + |w_+ - \overline{w}_+| < \delta$$

then

$$\text{meas}\{\xi \in \mathbb{R} \mid |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \overline{w}(\xi)| \geq \varepsilon\} < \gamma,$$

where $(\bar{u}(\xi), \overline{w}(\xi))$ *is the solution of* (1.1a, b) *with Riemann initial values* $(\bar{u}_-, \overline{w}_-)$ *and* $(\overline{u}_+, \overline{w}_+)$, *and* 'meas' *denotes the Lebesgue measure.*

2. The structure of solutions

Definition 2.1. A solution of (1.1) is said to be admissible according to the viscosity-capillarity travelling wave criterion (or travelling wave criterion for short) if

(i) at each point xi_0 of discontinuity of $(u(\xi), w(\xi))$, $(u(\xi_0-), w(\xi_0-))$ and $(u(\xi_0+), w(\xi_0+))$ exist, and

(ii) the following boundary value problem has a solution:

$$\frac{d^2\hat{w}(\zeta)}{d\zeta^2} = -2\xi_0 \frac{d\hat{w}(\zeta)}{d\zeta} + p(\xi_0-) - p(\hat{w}(\zeta)) - \xi_0^2(\hat{w}(\zeta) - w(\xi_0-)), \tag{2.1a}$$

$$\hat{w}(-\infty) = w(\xi_0-), \quad \hat{w}(+\infty) = w(\xi_0+), \quad \hat{w}'(\pm\infty) = 0. \tag{2.1b}$$

We first summarize our earlier results on the existence of solutions of (1.1) in the following theorem which is a combination of Corollary 4.7 of [8] and Theorem 4.3 of [7].

Theorem 2.1. (i) *Solutions of* (1.1), $(u(\xi), w(\xi))$, *which are admissible by the viscosity-capillarity travelling wave criterion exist which satisfy the condition that there is a* $\xi_0 \in \mathbb{R}$ *such that* $w(\xi) \leq \alpha$ *for* $\xi < \xi_0$ *and* $w(\xi) \geq \beta$ *for* $\xi > \xi_0$.

Lemma 2.2. *The boundary value problem* (2.1) *is equivalent to the following system*:

$$\frac{d\hat{u}(\zeta)}{d\zeta} = -s(\hat{u}(\zeta) - u_1) - p(\hat{w}(\zeta)) + p(w_1), \tag{2.2a}$$

$$\frac{d\hat{w}(\zeta)}{d\zeta} = -s(\hat{w}(\zeta) - w_1) - \hat{u}(\zeta) + u_1, \tag{2.2b}$$

$$(\hat{u}(-\infty), \hat{w}(-\infty)) = (u_1, w_1), \qquad (\hat{u}(\infty), \hat{w}(\infty)) = (u_2, w_2), \tag{2.2c}$$

where (u_1, w_1), (u_2, w_2) *satisfy the Rankine-Hugoniot condition at* s.

Proof. Obvious. □

In this paper, we always use the following notation:

$$\lambda(w) := \sqrt{-p'(w)}.$$

Lemma 2.3. *Let* $(u(\xi), w(\xi))$ *be a solution of* (1.1) *admissible by the travelling wave criterion. Then the following assertions hold*:

(i) $w(\xi) \notin (\alpha, \beta)$ *for any* $\xi \in \mathbb{R}$.

(ii) *If* ξ_0 *is a point of discontinuity of* $(u(\xi), w(\xi))$ *and* $w(\xi_0\pm) \leq \alpha$ (*or* $\geq \beta$), *then either*

$$\lambda(w(\xi_0-)) \geq \xi_0 \geq \lambda(w(\xi_0+)) \tag{2.3a}$$

or

$$-\lambda(w(\xi_0-)) \geq \xi_0 \geq -\lambda(w(\xi_0+)). \tag{2.3b}$$

(iii) *If there is a sequence* $\{\xi_n\}$ *such that* $\xi_n \to \xi_0-$ (*or* $\xi_n \to \xi_0+$), *as* $n \to \infty$, *and* $w(\xi_n+) \neq w(\xi_0-)$ (*or* $w(\xi_n-) \neq w(\xi_0+)$), *then*

$$\xi_0 = \pm\lambda(w(\xi_0-)) \quad (\text{ or } \xi_0 = \pm\lambda(w(\xi_0+))). \tag{2.4}$$

(iv) *If* $w(\xi_0-) \leq \alpha$ *and* $w(\xi_0+) \geq \beta$, *then*

$$\lambda(w(\xi_0-)) \geq \xi_0 \geq -\lambda(w(\xi_0+)). \tag{2.5}$$

Proof. (i) The proof is lengthy; therefore we put it in the Appendix.

(ii)–(iv) See the proofs of Proposition 3.6 of [3] and Theorems 5.1, 5.2 of [7], and apply Lemma 2.2. □

We define

$$f(w, w_1, s) := -p(w) + p(w_1) - s^2(w - w_1). \tag{2.6}$$

Lemma 2.4. *Suppose* (2.1) *has a solution.*

(i) *If* $\xi_0 \geq 0$, *then*

$$\int_{w_1}^{w} f(w, w_1, \xi_0)\, dw \geq 0 \quad \text{for } w \in [w_1, w_2]. \tag{2.7a}$$

(ii) *If* $\xi_0 < 0$, *then*

$$\int_{w}^{w_2} f(w, w_1, \xi_0)\, dw < 0 \quad \text{for } w \in [w_1, w_2]. \tag{2.7b}$$

Proof. We only prove (i) since the proof of (ii) is similar. Multiplying (2.1) by $\frac{d\hat{w}(\zeta)}{d\zeta}$ and integrating it on $(-\infty, \xi)$, we obtain

$$\int_{w_1}^{w} f(w, w_1, \xi_0)\, dw = \frac{1}{2}\left(\left.\frac{d\hat{w}(\zeta)}{d\zeta}\right|_{\zeta=\xi}\right)^2 + 2\xi_0 \int_{-\infty}^{\xi} \left(\frac{d\hat{w}(\zeta)}{d\zeta}\right)^2 d\zeta \geq 0. \tag{2.8}$$ □

In the remainder of this paper, we assume Assumption 1.

Theorem 2.5. *Let* ξ_0 *be a point of discontinuity of* $(u(\xi), w(\xi))$.

(i) *If* $w(\xi_0-) \leq \alpha$ *and* $w(\xi_0+) \geq \beta$, *then*

$$\lambda(w(\xi_0-)) > \xi_0 > -\lambda(w(\xi_0+)). \tag{2.9}$$

(ia) *If further* $w(\xi_0+) \neq w_+$, *then* $\xi_0 \leq \lambda(w(\xi_0+))$.

(ib) *If* $w(\xi_0-) \neq w_-$, *then* $\xi_0 \geq -\lambda(w(\xi_0-))$.

(ii) *If* $w(\xi_0\pm) \leq \alpha$, *then* $\xi_0 < 0$, $w(\xi_0+) < w(\xi_0-)$, *and*

$$-\lambda(w(\xi_0-)) > \xi_0 > -\lambda(w(\xi_0+)). \tag{2.10}$$

(iii) *If* $w(\xi_0\pm) \geq \beta$, *then* $\xi_0 > 0$, $w(\xi_0-) > w(\xi_0+)$, *and*

$$\lambda(w(\xi_0-)) > \xi_0 > \lambda(w(\xi_0+)). \tag{2.11}$$

Proof. (i) If

$$\xi_0 \geq \lambda(w(\xi_0-)), \tag{2.12}$$

then the chord connecting $w(\xi_0-)$ and $w(\xi_0+)$ lies below the graph of $p(w)$ which violates Lemma 2.4(i). Thus

$$\xi_0 < \lambda(w(\xi_0-)). \tag{2.13}$$

The other half of (2.11) can be proved similarly.

Let $w(\xi_0+) \neq w_+$ and suppose, for contradiction, that

$$\xi_0 > \lambda(w(\xi_0+)). \tag{2.14}$$

Then, by Lemma 2.3, $(u(\xi), w(\xi))$ is constant in (ξ_0, ξ_1) for some $\xi_1 > \xi_0$. We denote the supremum of such ξ_1 by ξ_2, and hence

$$\xi_2 > \xi_0 \tag{2.15}$$

and

$$w(\xi_2-) = w(\xi_0+). \tag{2.16}$$

If ξ_2 is a point of discontinuity of $(u(\xi), w(\xi))$, then $\xi_2 \leq \lambda(w(\xi_2-))$. If ξ_2 is a point of continuity, then, by the definition of ξ_2 and Lemma 2.3, $\xi_2 = \lambda(w(\xi_2))$. In both cases, we have

$$\xi_2 \leq \lambda(w(\xi_2-)) = \lambda(w(\xi_0+)) < \xi_0,$$

which contradicts (2.15).

The proof for (ib) is similar to that of (ia).

(ii) By Lemma 2.3, we have either

$$\lambda(w(\xi_0+)) \leq \xi_0 \leq \lambda(w(\xi_0-)), \tag{2.17a}$$

or

$$-\lambda(w(\xi_0+)) \leq \xi_0 \leq -\lambda(w(\xi_0-)). \tag{2.17b}$$

We claim that (2.17a) does not hold. Indeed, if otherwise, $\xi_0 > 0$. Since $w(\xi_0\pm) \leq \alpha$, there is an $\eta > \xi_0 > 0$ such that $w(\eta-) \leq \alpha < \beta \leq w(\eta+)$. It follows from (i) that

$$\eta < \lambda(w(\eta-)).$$

By Lemma 2.3(iii), $w(\xi)$ will be constant in (η_1, η). We let η_2 be the infimum of such η_1 and therefore

$$\eta > \eta_2 \geq \xi_0 > 0. \tag{2.18}$$

If η_2 is a point of discontinuity of $(u(\xi), w(\xi))$, then $w(\eta_2\pm) \leq \alpha$ and hence $\eta_2 \geq \lambda(\eta_2+)$. If η_2 is a point of continuity of $(u(\xi), w(\xi))$, then, by the definition of η_2, $\eta_2 = \lambda(w(\eta_2))$. In both cases, we have

$$\eta_2 \geq \lambda(w(\eta_2+)) = \lambda(w(\eta-)) > \eta,$$

which contradicts (2.18). Thus only (2.17b) holds. Equality in (2.17b) cannot hold since $p(w)$ is convex and

$$\xi_0^2 = -\frac{p(w(\xi_0+)) - p(w(\xi_0-))}{w(\xi_0+) - w(\xi_0-)}.$$

Therefore (2.10) is proved. $w(\xi_0+) < w(\xi_0-)$ follows easily from the convexity of $p(w)$ and (2.10).

(iii) The proof for (iii) is similar to that of (ii). □

Theorem 2.6. (i) *If there is a sequence* $\{\xi_n\}$ *such that* $\xi_n \to \xi_0-$, *as* $n \to \infty$, *and* $w(\xi_n+) \neq w(\xi_0-)$, *and*

(ia) *if* $w(\xi_0-) \leq \alpha$, *then*

$$\xi_0 = -\lambda(w(\xi_0-)); \tag{2.19a}$$

(ib) *if* $w(\xi_0-) \geq \beta$, *then*

$$\xi_0 = \lambda(w(\xi_0-)). \tag{2.19b}$$

(ii) *If there is a sequence* $\{\xi_n\}$ *such that* $\xi_n \to \xi_0+$, *as* $n \to \infty$, *and* $w(\xi_n-) \neq w(\xi_0+)$, *and*

(iia) *if* $w(\xi_0+) \leq \alpha$, *then*

$$\xi_0 = -\lambda(w(\xi_0+)); \tag{2.20a}$$

(iib) *if* $w(\xi_0+) \geq \beta$, *then*

$$\xi_0 = \lambda(w(\xi_0+)). \tag{2.20b}$$

Proof. We prove (ia) only since the proofs for the rest of the theorem are similar.

In this case, $\xi_0 = \pm\lambda(w(\xi_0-))$ as asserted by Lemma 2.3. Assume, for contradiction, $w(\xi_0-) \leq \alpha$ and $\xi_0 > 0$. Then there is an $\eta \in \mathbb{R}$ such that $\eta \geq \xi_0$ and $w(\eta-) \leq \alpha < \beta \leq w(\eta+)$. By Theorem 2.5(i), $\eta = \xi_0$ cannot hold and hence $\eta > \xi_0$. By Lemma 2.3, $w(\xi)$ will be constant in (η_1, η) for some $\eta_1 < \eta$. Similar to what we did in the proof of Theorem 2.5(ii), we let η_2 be the infimum of such η_1 and therefore

$$\eta > \eta_2 \geq \xi_0 > 0. \tag{2.21}$$

Theorem 2.5(ii) says that η_2 cannot be a point of discontinuity of $(u(\xi), w(\xi))$ or otherwise $\eta_2 < 0$ which is prohibited by (2.21). Thus η_2 is a point of continuity of $(u(\xi), w(\xi))$. By the definition of η_2, we can see that $\eta_2 = \lambda(w(\eta_2))$. Therefore,

$$\eta_2 = \lambda(w(\eta_2+)) = \lambda(w(\eta-)) > \eta,$$

which contradicts (2.21). □

Theorem 2.7. *Let* $(u(\xi), w(\xi))$ *be a solution of* (1.1).

(i) *There is one and only one phase boundary in the solution, i.e., there is a* $\xi_0 \in \mathbb{R}$ *such that* $w(\xi) > \beta$ *for* $\xi > \xi_0$, *and* $w(\xi) < \alpha$ *for* $\xi < \xi_0$

(ii) *In the region* $w \leq \alpha$, *solutions of* (1.1), $(u(\xi), w(\xi))$, *consist of either a constant state* (u_-, w_-) *or two constant states* (u_-, w_-) *and* (u_1, w_1) *joined by a shock with speed* $s_1 < 0$ *or a backward rarefaction wave with* (u_-, w_-) *on its left.*

(iii) *In the region* $w \geq \beta$, *solutions of* (1.1), $(u(\xi), w(\xi))$, *consist of either a constant state* (u_+, w_+) *or two constant states* (u_+, w_+) *and* (u_2, w_2) *joined by a shock with speed* $s_3 > 0$ *or a forward rarefaction wave with* (u_+, w_+) *on its right.*

(iv) (u_1, w_1) *and* (u_2, w_2) *are joined by a shock, i.e., the phase boundary.*

Proof. We prove (i) and (ii) only since that of (iii) is similar, and (iv) follows immediately.

(i) Suppose that there are more than one phase boundaries in a solution $(u(\xi), w(\xi))$. Then there are at least three phase boundaries because $w_- < \alpha < \beta < w_+$. More precisely, there are points of discontinuity ξ_0, ξ_1, $\xi_2 \in \mathbb{R}$

of $(u(\xi), w(\xi))$ such that $w(\xi_j-) < \alpha < \beta < w(\xi_j+)$, $j = 0, 2$, and $w(\xi_1+) < \alpha < \beta < w(\xi_1-)$. Without loss of generality, we assume $\xi_0 < \xi_1 < \xi_2$ and that there are no other points of dicontinuity of $(u(\xi), w(\xi))$ between ξ_0 and ξ_2. At least two of ξ_j, $j = 0, 1, 2$, are nonnegative or nonpositive. We consider the case $0 \le \xi_1 < \xi_2$ only, since the proof for the other cases are similar. For the point ξ_1, we know from (2.8) in the proof of Lemma 2.4 that

$$\int_{w(\xi_1+)}^{w(\xi_1-)} f(w, w(\xi_1+), \xi_1)\, dw \le 0. \tag{2.22}$$

Theorems 2.5(ii) and 2.6(ii) imply that $(u(\xi), w(\xi))$ is constant for $\xi \in (\xi_1, \xi_2)$ since $\xi_1 \ge 0$. Thus $w(\xi_2-) = w(\xi_1+)$. Lemma 2.4 then leads to

$$\int_{w(\xi_1+)}^{w(\xi_2+)} f(w, w(\xi_1+), \xi_2)\, dw > 0. \tag{2.23}$$

From (2.22) and (2.23), we know, by an inspection on the graph of $p(w)$, that $\xi_1 > \xi_2 > 0$, which is a contradiction.

(ii) Suppose $(u(\xi), w(\xi))$ has a point ξ_0 of discontinuity with $w(\xi_0\pm) \le \alpha$. By Theorem 2.5, $\xi_0 < 0$. We define a subset of $\mathbb{R}$ by

$$A := \{\xi > \xi_0 \mid w(\xi) \ne w(\xi_0+),\, w(\xi\pm) \le \alpha\}. \tag{2.24}$$

We claim that A is empty. Indeed, if otherwise, we can define

$$\eta := \inf A \ge \xi_0. \tag{2.25}$$

If $\eta = \xi_0$, then there is a sequence $\xi_n \to \eta+ = \xi_0+$ such that $w(\xi_n+) \ne w(\xi_0+)$. By Theorem 2.6,

$$\xi_0 = -\lambda(w(\xi_0+)),$$

which is impossible in view of Theorem 2.5. If $\eta > \xi_0$, then $w(\eta-) = w(\xi_0+)$. If, further, $w(\eta-) \ne w(\eta+)$, then, by Theorem 2.5,

$$\eta < -\lambda(w(\eta-)) = -\lambda(w(\xi_0+)) < \xi_0,$$

which contradicts the definition of η in (2.25). If $w(\eta-) = w(\eta+)$, then there is a sequence $\xi_n \to \eta+$ as $n \to \infty$ such that $w(\xi_n+) \ne w(\eta+)$. Again by Theorem 2.6, we obtain

$$\eta = -\lambda(w(\eta\pm)) = -\lambda(w(\xi_0+)) < \xi_0,$$

which is also impossible. Thus, A is empty, which simply says that $(u(\xi), w(\xi))$ must be constant for $\xi \in (\xi_0, s_2)$ for some s_2 such that $w(s_2+) \ge \beta$.

Similarly, we can prove that $(u(\xi), w(\xi))$ is constant for $\xi \in (-\infty, \xi_0)$. □

From the above results, we can see that a solution $(u(\xi), w(\xi))$ of (1.1) consists of a shock $\xi = s_2$, such that $w_1 := w(s_2-) \le \alpha < \beta \le w(s_2+) =: w_2$, and two constant states (u_-, w_-) and (u_+, w_+). (u_-, w_-) is joined to $(u(s_2-), w(s_2-))$ by either a backward shock $\xi = s_1 < 0$ if $w(s_2-) < w_-$ or a backward rarefaction wave if $w(s_2-) > w_-$. (u_+, w_+) is connected to $(u(s_2+), w(s_2+))$ by either a forward shock $\xi = s_3 > 0$ if $w(s_2+) > w_+$ or a forward rarefaction wave if $w(s_2+) < w_+$. Thus, we can denote a solution of (1.1), for simplicity, by $\{w_1, w_2, s_2\}$.

3. The phase boundary

Let $\xi = s$ be a point of discontinuity of $(u(\xi), w(\xi))$. For notational simplicity, we denote $(u_1, w_1) := (u(s-), w(s-))$, $(u_2, w_2) := (u(s+), w(s+))$. Then we have the Rankine-Hugoniot conditions:

$$-s(u_2 - u_1) + p(w_2) - p(w_1) = 0, \tag{3.1}$$

$$-s(w_2 - w_1) - (u_2 - u_1) = 0. \tag{3.2}$$

The speed s of the shock is determined by

$$s^2 = -\frac{p(w_2) - p(w_1)}{w_2 - w_1}. \tag{3.3}$$

We call $w_1 \to w_2$ a connection if (2.1) has a solution. In this section, we devote ourselves to connections $w_1 \to w_2$, where $w_1 \le \alpha < \beta \le w_2$, which are called phase boundaries.

Lemma 3.1. *Let $\alpha > w_1 \to w_2 > \beta$ be a connection with speed s.*

(i) *If $s = 0$, then $w_1 = m$ and $w_2 = M$, where $m < \alpha$ and $M > \beta$ are Maxwell constants defined by*

$$p(m) = p(M), \qquad \int_m^M (p(w) - p(m))\, dw = 0. \tag{3.4}$$

(ii) *If $s \ge 0$, then $w_1 \le m$.*

(iii) *If $s < 0$, then $w_2 > M$.*

Proof. (i) is proved in [15]. (ii) and (iii) follow from Lemma 2.4. □

Lemma 3.2. *Let $s \ge 0$ in (2.1). Then the connected component of the unstable manifold of $(w_1, 0)$ in the upper half phase plane containing $(w_1, 0)$ is unique and is denoted by $\Gamma_1(w_1, w_2)$.*

The connected component of the stable manifold of $(w_2, 0)$ in the upper half phase plane containing $(w_2, 0)$ is unique and is denoted by $\Gamma_2(w_1, w_2)$.

Proof. We can rewrite (2.1) as

$$v\frac{dv}{d\hat{w}} = -2sv + f(\hat{w}, w_1, s), \tag{3.5a}$$

$$\frac{d\hat{w}}{d\zeta} = v, \tag{3.5b}$$

where $f(w, w_1, s) := -p(w) + p(w_1) - s^2(w - w_1)$. Along the trajectories of (3.5) which are in the upper half phase plane, $\hat{w}'(\zeta) = v \ge 0$. Thus, we can parametrize this part of the trajectories by $v(w)$.

Suppose (3.5) has two unstable manifolds leaving $(w_1, 0)$ into the upper half phase plane. We denote these two manifolds by $v(w)$ and $\bar{v}(w)$ respectively. If $v(w_0) = \bar{v}(w_0) > 0$ for some $w_0 > w_1$, then the uniqueness of (3.5) in half plane $v > 0$ implies $v(w) = \bar{v}(w)$. Thus, without loss of generality, we can assume

$$v(w) > \bar{v}(w) \tag{3.6}$$

for $w \in (w_1, w_1 + \mu)$ for some $\mu > 0$. Then it follows from (3.5) that

$$v \frac{dv}{dw} - \bar{v} \frac{d\bar{v}}{dw} = -2s(v - \bar{v}). \tag{3.7}$$

Integrating (3.7) from w_1 to w, we obtain, by virtue of (3.6) and $s \geq 0$, that

$$0 < v^2(w) - \bar{v}^2(w) = -2s \int_{w_1}^{w} (v(w) - \bar{v}(w))\, dw \leq 0$$

for $w \in (w_1, w_1 + \mu)$, which is the desired contradiction.

The proof for the uniqueness of the stable manifold entering $(w_2, 0)$ is similar. □

Lemma 3.3. *Let* $w_1 \leq \alpha$, $w_2 \geq \beta$ *satisfy*

$$s \leq \lambda(w_2), \tag{3.8}$$

where s^2 *is determined by* (3.2). *Then* $w_1 \to w_2$ *is a connection with* $s \geq 0$ *if and only if* $\Gamma_1(w_1, w_2) = \Gamma_2(w_1, w_2)$.

Proof. It is clear that if $\Gamma_1(w_1, w_2) = \Gamma_2(w_1, w_2)$, then $w_1 \to w_2$ is a connection.

Conversely, if $w_1 \to w_2$ is a connection, then the connecting trajectory $\hat{w}(\zeta)$ has to cross the w-axis at some $(w_0, 0)$ for some $w_0 \geq w_2$. On the other hand (3.8) and Assumption 1 together with (2.1) imply that $w_0 \leq w_2$. Thus $\Gamma_1(w_1, w_2) = \Gamma_2(w_1, w_2)$. □

We denote the manifolds $\Gamma_1(w_1, w_2)$ and $\Gamma_2(w_1, w_2)$ by $v_1(w, s)$ and $v_2(w, s)$ respectively to specify the dependence of v_1, v_2 on s.

The proof of the following lemma was first given by M. Shearer [16]:

Lemma 3.4. (i)

$$\frac{\partial v_1}{\partial s}(w, s) < 0.$$

(ii)

$$\frac{\partial v_2}{\partial s}(w, s) > 0.$$

Proof. (i) It follows from (3.5) that

$$\frac{dv}{dw} = -2s + \frac{f(w, w_1, s)}{v}. \tag{3.9}$$

As $w \to w_1$ or w_2, $f(w, w_1, s) \to 0$ and $v \to 0$. A simple computation based on (3.9) shows that

$$\left.\frac{\partial v}{\partial w}\right|_{w=w_1} = -s + \lambda(w_1), \tag{3.10}$$

and hence

$$\left.\frac{\partial^2 v_1}{\partial s \partial w}\right|_{w=w_1} = -1. \tag{3.11}$$

Since $v_1(w_1) \equiv 0$ and thus $\frac{\partial v_1}{\partial s}|_{w=w_1} = 0$,

$$\frac{\partial v_1}{\partial s}(w, s) < 0 \tag{3.12}$$

for $w > w_1$ and near w_1.

On the other hand, by integrating (3.5) from w_1 to some w, we obtain

$$\frac{v_1^2(w)}{2} = \int_{w_1}^{w} [-2sv_1(\theta) + f(\theta, w_1, s)]\,d\theta, \tag{3.13}$$

(3.14)
$$v_1(w)\frac{\partial v_1}{\partial s}(w, s) = 2\int_{w_1}^{w}\left[-v_1(\theta) - s\frac{\partial v_1}{\partial s}(\theta, s) - s(\theta - w_1)\right] d\theta \equiv 2g(w).$$

Now, we claim $\frac{\partial v_1}{\partial s}(w, s) < 0$. Indeed, if otherwise, there is $w^* > w_1$ such that $\frac{\partial v_1}{\partial s}(w^*, s) = 0$ and $\frac{\partial v_1}{\partial s}(w, s) < 0$ for all $w \in (w_1, w^*)$. In other words, $g(w_1) = g(w^*) = 0$ and $g(w) < 0$ for all $w \in (w_1, w^*)$. Thus

$$g'(w^*) \geq 0. \tag{3.15}$$

However,

$$g'(w^*) = -v_1(w^*) - s(w^* - w_1) < 0, \tag{3.16}$$

which contradicts (3.15). Thus $\frac{\partial v_1}{\partial s}(w, s) < 0$.

(ii) The proof is similar to that of (i). □

We define

$$w_2(w_1, s) := \max\{w \geq \beta \mid p(w) = p(w_1) - s^2(w - w_1)\}, \tag{3.17a}$$

$$w_3(w_1, s) := \min\{w \geq \beta \mid p(w) = p(w_1) - s^2(w - w_1)\}. \tag{3.17b}$$

Theorem 3.5. *Let $\alpha > w_1 \to w_2 > \beta$ be a connection with speed $\lambda(w_2) > s \geq 0$. Then any $w^* \in [w_3(w_1, s), w_2)$ cannot be connected to w_1.*

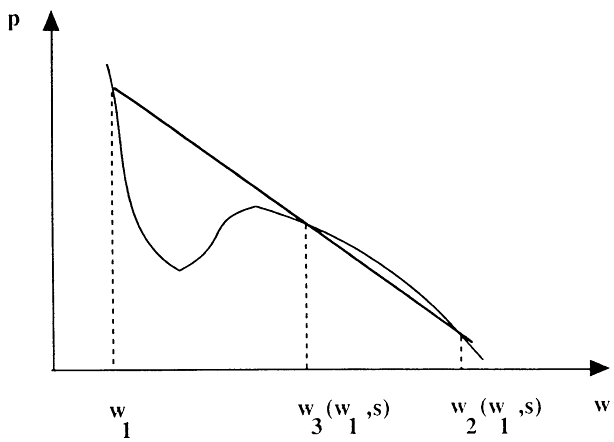

FIGURE 2

Proof. Assume the contrary, i.e. there is a $w^* \in [w_3(w_1, s), w_2)$ such that $w_1 \to w^*$ is a connection. An inspection on the graph of $p(w)$ tells us that

$$s^{*2} = -\frac{p(w^*) - p(w_1)}{w^* - w_1} < s^2. \tag{3.18}$$

By Lemma 2.4, $0 \le s^* < s$. Lemma 3.3 implies that

$$v_1(w, s^*) > v_1(w, s) > 0 \tag{3.19}$$

for $w \in [w_3(w_1, s), w_2)$. It is clear from (3.18) that $f(w, w_1, s^*) > 0$ for $w \ge w_2$ and hence $v_1(w, s_3) > 0$ for $w \ge w_0(s)$. That is, $w_1 \to w^*$ is not a connection. □

Theorem 3.6. *Let $\alpha > w_1 \to w_2 > \beta$ be a connection with speed $0 \le s \le \lambda(w_2)$. If $\alpha > \overline{w}_1 \to \overline{w}_2 > \beta$ is a connection with speed $\overline{s} \ge 0$ and if $\overline{w}_1 < w_1$, then $\overline{s} > s$.*

Proof. Consider the unstable manifold $\Gamma_1(\overline{w}_1, \overline{w}_2)$ leaving $(\overline{w}_1, 0)$ into the upper half phase plane which we shall denote by $\bar{v}(w)$. We have

$$\bar{v}\frac{d\bar{v}}{dw} = -2\overline{s}\,\overline{w} - p(w) + p(\overline{w}_1) - \overline{s}^2(w - \overline{w}_1). \tag{3.20}$$

Since $\overline{w}_1 \to \overline{w}_2$ is a connection, $\Gamma_1(\overline{w}_1, \overline{w}_2)$ must cross the w-axis at $(w_0, 0)$ for some $w_0 \le w_2(\overline{w}_1, \overline{s})$, where the unstable manifold $\Gamma_1(w_1, w_2(w_1, \overline{s}))$, parametrized by $v(w)$ satisfies

$$v\frac{dv}{dw} = -2\overline{s}v - p(w) + p(w_1) - \overline{s}^2(w - w_1). \tag{3.21}$$

Subtracting (3.20) from (3.21), we obtain

$$v\frac{dv}{dw} - \bar{v}\frac{d\bar{v}}{dw} = -2\overline{s}(v - \bar{v}) + p(w_1) - p(\overline{w}_1) - \overline{s}^2(\overline{w}_1 - w_1). \tag{3.22}$$

If $\overline{s} \ge \lambda(w_1)$, then there is nothing to be proved since $s < \lambda(w_1)$ by Theorem 2.5. Thus, without loss of generality, we assume $\overline{s}^2 < -p'(w_1)$. Then

$$\overline{s}^2 < \min(-p'(\overline{w}_1), -p'(w_1)) < -\frac{p(\overline{w}_1) - p(w_1)}{\overline{w}_1 - w_1}. \tag{3.23}$$

The last inequality comes from the convexity of $p(w)$ in the region $w < \alpha$. Applying (3.23) to (3.22), we get

$$v\frac{dv}{dw} - \bar{v}\frac{d\bar{v}}{dw} < -2\overline{s}(v - \bar{v}). \tag{3.24}$$

Now, we claim that $v(w)$, $\bar{v}(w)$ do not intersect. Since $\bar{v}(w_1) > v(w_1) = 0$, if the contrary of our claim holds, there will be $w^* > \overline{w}_1$ such that $\bar{v}(w) > v(w)$ for $w \in [w_1, w^*)$ and $v(w^*) = \bar{v}(w^*)$. Thus, $v'(w^*) \ge \bar{v}'(w^*)$, which contradicts the consequence of (3.24)

$$v(w^*)\frac{d(v - \bar{v})}{dw}\bigg|_{w=w^*} < 0 \tag{3.25}$$

and hence our claim. Therefore, $v(w)$ must meet the w-axis at some $(w_0, 0)$, where $w_0 \in (w_1, w_2(\overline{w}_1, \overline{s}))$. If $s \ge \overline{s}$, then, by Lemma 3.3, $\Gamma_1(\overline{w}_1, \overline{w}_2)$ will meet the v-axis at some $w \in (w_1, w_2(\overline{w}_1, \overline{s}))$ and hence cannot join $w_2(w_1, s) > w_2(\overline{w}_1, \overline{s})$. This simply says that $w_1 \to w_2$ is not a connection with $0 \le s \le \lambda(w_2)$. This contradiction completes our proof. □

For connections $\alpha > w_1 \to w_2 > \beta$ with $s \le 0$, we have the following similar results.

Theorem 3.7. (i) *Let* $w_1 \to w_2$ *be a connection with* $-\lambda(w_1) \le s \le 0$. *Then any* $w^* \in (w_1, w_0(w_2, s))$ *cannot be connected to* w_2, *where*

$$w_0(w_2, s) := \max\{w < \alpha \mid p(w) = p(w_2) - s^2(w - w_2)\}.$$

(ii) *Let* $\alpha > w_1 \to w_2 > \beta$ *be a connection with speed* $0 \ge s \ge -\lambda(w_2)$. *If* $\alpha > \overline{w}_1 \to \overline{w}_2 > \beta$ *is a connection with speed* $\overline{s} \le 0$ *and if* $\overline{w}_2 > w_2$, *then* $\overline{s} < s$.

Proof. Assume $w_1 \to w_2$ is a connection with $s \le 0$. Then (2.1) has a solution. We can replace w_1 in (2.1) by w_2, since we have the Rankine-Hugoniot conditions (3.1), to obtain

$$\begin{gathered}\frac{d^2\hat{w}}{d\zeta^2} = -2s\frac{d\hat{w}(\zeta)}{d\zeta} - p(\hat{w}) + p(w_2) - s^2(\hat{w} - w_2), \\ \hat{w}(-\infty) = w_1, \quad \hat{w}(+\infty) = w_2, \quad \hat{w}'(\pm\infty) = 0.\end{gathered} \tag{3.26}$$

After applying the following transformations

$$\zeta \mapsto -\zeta, \qquad \hat{w} \mapsto -\hat{w} \tag{3.27}$$

in (2.1), (3.26) becomes

$$\begin{gathered}\frac{d^2\hat{w}}{d\zeta^2} = -2(-s)\frac{d\hat{w}(\zeta)}{d\zeta} - P(\hat{w}) + P(-w_2) - s^2(\hat{w} - (-w_2)), \\ \hat{w}(-\infty) = -w_2, \quad \hat{w}(+\infty) = -w_1, \quad \hat{w}'(\pm\infty) = 0,\end{gathered} \tag{3.28}$$

where $P(w) := -p(-w)$ also satisfies Assumption 1. Now we can apply Theorems 3.5, 3.6 to (3.28) to complete the proof of our theorem. □

4. The uniqueness and stability of the solution of (1.1)

Lemma 4.1. (1.1) *has a solution* $\{w_1, w_2, s_2\}$ *if and only if the following conditions hold*:

$$F(w_1, w_2, s_2) = u_+ - u_-, \tag{4.1}$$

where $w_1 \to w_2$ *is a connection with speed* s_2 *and* $w_1 \le \alpha < \beta \le w_2$,

$$\begin{aligned}F(w_1, w_2, s_2) := & - s_1(w_1 - w_-)H(w_- - w_1) \\ & + H(w_1 - w_-+)\int_{w_-}^{w_1} \lambda(w)\,dw \\ & - s_2(w_2 - w_1) - H(w_2 - w_+)s_3(w_+ - w_2) \\ & + H(w_+ - w_2+)\int_{w_+}^{w_2} \lambda(w)\,dw,\end{aligned} \tag{4.2}$$

where $H(w)$ *is the Heaviside function and*

$$s_k^2 := -\frac{p(w_k) - p(w_{k-1})}{w_k - w_{k-1}}, \qquad k = 1, 2, 3, \tag{4.3}$$

$w_0 := w_-$, $w_3 := w_+$;

$$s_1 < 0, \quad s_3 > 0, \quad s_1 < s_2 < s_3, \tag{4.4a}$$

$$-\lambda(w_1) < s_1 < -\lambda(w_-), \qquad \lambda(w_+) < s_3 < \lambda(w_2), \tag{4.4b}$$

(4.4c) $$-\lambda(w_2) < s_2 < \lambda(w_1),$$

(4.4d) $$s_2 \le \lambda(w_2) \quad \text{if } w_2 \ne w_+,$$

(4.4e) $$s_2 \ge -\lambda(w_1) \quad \text{if } w_1 \ne w_-.$$

Proof. Suppose (1.1) has a solution, denoted by $\{w_1, w_2, s_2\}$. From the discussion following Theorem 2.7, $\{w_1, w_2, s_2\}$ has the following structure: When $w_1 := w(s_2-) < w_-$, the solution $(u(\xi), w(\xi))$, for $\xi < s_2$, consists of a shock of speed $s_1 < 0$ joining constant states (u_-, w_-) and (u_1, w_1). When $w_1 \ge w_-$, the solution $(u(\xi), w(\xi))$, for $\xi < s_2$, consists of a backward rarefaction wave connecting (u_-, w_-) and (u_1, w_1). Therefore,

(4.5) $$u_1 - u_- = -H(w_- - w_1)s_1(w_1 - w_-) + H(w_1 - w_-)\int_{w_-}^{w_1} \lambda(w)dw,$$

where

(4.6) $$s_1 = -\frac{p(w_1) - p(w_-)}{w_1 - w_-}.$$

Similarly, we have

(4.7) $$u_2 - u_1 = -s_2(w_2 - w_1)$$

and

(4.8) $$u_+ - u_2 = -H(w_2 - w_+)s_3(w_+ - w_2) + H(w_+ - w_2)\int_{w_+}^{w_2} \lambda(w)\,dw,$$

where

(4.9) $$s_k = -\frac{p(w_k) - p(w_{k-1})}{w_k - w_{k-1}},$$

$k = 2, 3$, and $w_3 := w_+$. (4.2) follows easily from (4.5), (4.7), and (4.9). The constraints (4.4) follow from Theorems 2.5 and Lemma 2.4. The necessity is proved.

By Theorem 3.3 of [8], the shock solution $(u_-, w_-), (u_1, w_1)$ $((u_2, w_2), (u_+, w_+))$ is admissible if $w_1 < w_-$ $(w_2 > w_+)$. Thus, conditions in Lemma 4.1 simply say that $\{w_1, w_2, s_2\}$ satisfies the initial conditions (1.1c, d), is admissible, and hence is a solution of (1.1). Thus the sufficiency is also proved. □

In view of Theorems 3.4 and 3.8, we can also write $F(w_1, w_2, s_2)$ as $G(w_1, s_2)$ for $s_2 \ge 0$ and $J(w_2, s_2)$ for the case $s_2 \le 0$.

Remark 4.1. When we consider (4.1) as a necessary condition for (1.1) to have a solution, it will be more convenient to extend the domain of definition of $G(w_1, s_2)$ $(J(w_2, s_2))$ to include $s_2 = s_3$ $(s_2 = s_1)$. If a solution $\{w_1, w_2, s_2\}$ satisfies $0 \le s_2 \le \lambda(w_2)$, then $w_2 = w_2(w_1, s_2)$. If $s_2 > \lambda(w_2)$, then $w_2 = w_+ = w_3(w_1, s_2)$ and

$$u_+ - u_- = F(w_1, w_+, s_2) = F(w_1, w_2(w_1, s_2), s_2).$$

Thus, we can always take the w_2 in expression (4.2) for $G(w_1, s_2)$ to be $w_2(w_1, s_2)$. Therefore, we can always assume $0 \le s_2 \le \lambda(w_2)$ in $G(w_1, s_2)$ since $s \le \lambda(w_2(w_1, s))$ due to the concavity of $p(w)$ in the region $w \ge \beta$. Similar things can be said for $J(w_2, s_2)$.

Lemma 4.2. (i) *If* $0 \le s_2$, *then*

$$\frac{\partial G}{\partial w_1}(w_1, s_2) \ge \lambda(w_1) + s_2. \tag{4.10a}$$

(ii) *If* $0 \le s_2 \le \lambda(w_2)$, *then*

$$\frac{\partial G}{\partial s_2}(w_1, s_2) \le 0. \tag{4.10b}$$

The equality holds only if $s_2 = \lambda(w_2)$.

Proof. In this proof, we treat w_1, s_2 as variables. A straightforward computation based on (4.3) shows that

$$\frac{\partial s_1}{\partial w_1} = \frac{\lambda^2(w_1) - s_1^2}{2s_1(w_1 - w_-)}, \tag{4.11}$$

$$\frac{\partial w_2}{\partial w_1} = \frac{\lambda^2(w_1) - s_2^2}{\lambda^2(w_2) - s_2^2}, \tag{4.12}$$

$$\frac{\partial w_2}{\partial s_2} = \frac{2s_2(w_2 - w_1)}{\lambda^2(w_2) - s_2^2}, \tag{4.13}$$

$$\frac{\partial s_3}{\partial w_2} = \frac{s_3^2 - \lambda^2(w_2)}{2s_3(w_+ - w_2)}. \tag{4.14}$$

With these preparations, we can compute $\frac{\partial G}{\partial w_1}$ as follows:

(4.15)

$$\begin{aligned}
\frac{\partial G}{\partial w_1} &= H(w_- - w_1)\left[-(w_1 - w_-)\frac{\partial s_1}{\partial w_1} - s_1\right] + H(w_1 - w_-+)\lambda(w_1) + s_2 \\
&\quad + \frac{\partial w_2}{\partial w_1}\Bigg[-s_2 - H(w_2 - w_+)\left((w_+ - w_2)\frac{\partial s_3}{\partial w_2} - s_3\right) \\
&\qquad\qquad + H(w_+ - w_2+)\lambda(w_2)\Bigg] \\
&= H(w_- - w_1)\frac{\lambda^2(w_1) - 2s_1s_2 + s_1^2}{-2s_1} + H(w_1 - w_-+)(\lambda(w_1) + s_2) \\
&\quad + \frac{\lambda^2(w_1) - s_2^2}{\lambda^2(w_2) - s_2^2}\Bigg[H(w_2 - w_+)\frac{\lambda^2(w_2) - 2s_2s_3 + s_3^2}{2s_3} \\
&\qquad\qquad + H(w_+ - w_2+)(\lambda(w_2) - s_2)\Bigg].
\end{aligned}$$

Applying the inequality $\lambda^2(w_1) + s_1^2 \ge -2s_1\lambda(w_1)$, $s_1 < 0$, we can prove that

$$H(w_- - w_1)\frac{\lambda^2(w_1) - 2s_1s_2 + s_1^2}{-2s_1} + H(w_1 - w_-+)(\lambda(w_1) + s_2) \ge \lambda(w_1) + s_2.$$

Recalling constraints (4.4), we can see easily that the sum of the last two terms on the right-hand side of (4.15) is positive. This completes the proof of (i).

(ii) After a computation similar to those in (i), we obtain

$$\frac{\partial G}{\partial s_2} = \frac{w_1 - w_2}{\lambda(w_2)^2 - s_2^2}\left[H(w_2 - w_+)\frac{1}{s_3}(s_3 - s_2)(\lambda(w_2)^2 - s_2 s_3) + H(w_+ - w_2+)(\lambda(w_2) - s_2)^2\right]. \tag{4.16}$$

An analysis similar to that for (4.15) yields our result. □

Lemma 4.3. (i) *If* $s_2 \leq 0$, *then*

$$\frac{\partial J}{\partial w_2}(w_2, s_2) > \lambda(w_2) - s_2. \tag{4.17a}$$

(ii) *If* $0 \geq s_2 \geq \lambda(w_1)$, *then*

$$\frac{\partial J}{\partial s_2}(w_2, s_2) \leq 0. \tag{4.17b}$$

The equality holds only if $s_2 = -\lambda(w_1)$.

Proof. The proof is almost the same as that of Lemma 4.2. For completeness, we list some of the intermediate results in the following:

$$\frac{\partial J}{\partial w_1} = \frac{\lambda^2(w_2) - s_2^2}{\lambda^2(w_1) - s_2^2}\left[H(w_- - w_1)\frac{\lambda^2(w_1) - 2s_1 s_2 + s_1^2}{-2s_1} + H(w_1 - w_-+)(\lambda(w_1) + s_2)\right] + H(w_2 - w_+)\frac{\lambda^2(w_2) - 2s_2 s_3 + s_3^2}{2s_3} + H(w_+ - w_2+)(\lambda(w_2) - s_2),$$

$$\frac{\partial J}{\partial s_2} = \frac{w_1 - w_2}{\lambda(w_1)^2 - s_2^2}\left[H(w_- - w_1)\frac{1}{s_1}(s_1 - s_2)(\lambda(w_1)^2 - s_2 s_1) + H(w_1 - w_-+)(\lambda(w_1) + s_2)^2\right]. \quad \square$$

Lemma 4.4. (i) (1.1) *has at most one solution* $\{w_1, w_2, s_2\}$ *with* $s_2 \geq 0$.

(ii) (1.1) *has at most one solution* $\{w_1, w_2, s_2\}$ *with* $s_2 \leq 0$.

Proof. (i) We claim that (1.1) can have at most one solution $\{w_1, w_2, s_2\}$ with

$$0 \leq s_2 \leq \lambda(w_2). \tag{4.18}$$

Indeed, if we have two solutions of (1.1), $\{w_1, w_2, s_2\}$ and $\{\overline{w}_1, \overline{w}_2, \overline{s}_2\}$, with (4.18) and

$$0 \leq \overline{s}_2 \leq \lambda(\overline{w}_2) \tag{4.19}$$

and $(w_1, s_2) \neq (\overline{w}_1, \overline{s}_2)$, then we have, by Lemma 4.1,

$$G(w_1, s_2) = G(\overline{w}_1, \overline{s}_2) = u_+ - u_-. \tag{4.20}$$

(4.20) shows, with help from Lemma 4.2(i), that $w_1 \neq \overline{w}_1$. Without loss of generality, we assume that

$$\overline{w}_1 > w_1. \tag{4.21}$$

Then, by Theorem 3.6,

$$\overline{s}_2 < s_2. \tag{4.22}$$

Consider the difference

$$\begin{aligned} &G(\overline{w}_1, \overline{s}_2) - G(w_1, s_2) \\ &\quad = G(\overline{w}_1, \overline{s}_2) - G(\overline{w}_1, s_2) + G(\overline{w}_1, s_2) - G(w_1, s_2) \\ &\quad = \frac{\partial G}{\partial w_1^*}(w_1^*, s_2)\Big|_{w_1^* = w_0 \in (w_1, \overline{w}_1)} (\overline{w}_1 - w_1) \\ &\qquad + \frac{\partial G}{\partial s_2^*}(\overline{w}_1, s_2^*)\Big|_{s_2^* = s_0 \in (\overline{s}_2, s_2)} (\overline{s}_2 - s_2). \end{aligned} \tag{4.23}$$

Applying Remark 4.1, (4.21), (4.22), and Lemma 4.2, we find that

$$G(\overline{w}_1, \overline{s}_2) > G(w_1, s_2) = u_+ - u_-, \tag{4.24}$$

which contradicts (4.20). Thus, our claim holds.

Now, we claim that (1.1) has at most one solution $\{w_1, w_2, s_2\}$ with

$$s_2 > \lambda(w_2). \tag{4.25}$$

In this case, $w_2 = w_+$, and hence it is necessary, in view of Lemma 4.1, that

$$F(w_1, w_+, s_2) = u_+ - u_-. \tag{4.26}$$

After some computation, we obtain

$$\begin{aligned} \frac{dF}{dw_1} &= H(w_- - w_1)\frac{1}{2}(s_2 - s_1)\left(1 - \frac{\lambda(w_1)^2}{s_2 s_1}\right) \\ &\quad + H(w_1 - w_-+)\frac{(\lambda(w_1) + s_2)^2}{2s_2} > 0 \end{aligned} \tag{4.27}$$

if (4.25) holds. Thus, (4.26) has at most one solution w_1 and our claim is proven.

It remains to show that (1.1) cannot have a solution satisfying (4.18) and a solution satisfying (4.25) simultaneously. Suppose, for contradiction, that (1.1) has a solution $\{w_1, w_2, s_2\}$ satisfying (4.18) and a solution $\{\overline{w}_1, \overline{w}_2, \overline{s}_2\}$ satisfying (4.25) and hence $\overline{w}_2 = w_+$.

We define

$$w_2(w_1, s) := \max\{w \geq \beta \mid p(w) = p(w_1) - s^2(w - w_1)\}, \tag{4.28a}$$

$$w_3(w_1, s) := \min\{w \geq \beta \mid p(w) = p(w_1) - s^2(w - w_1)\}. \tag{4.28b}$$

Case 1. $w_1 > \overline{w}_1$. In this case, by Theorem 3.6,

$$\overline{s}_2 > s_2. \tag{4.29}$$

If, further, $w_3(w_1, s_2) > w_+$, then an inspection of the graph of $p(w)$ will tell us that $s_3 < s_2$, which is unacceptable by (4.4). If, on the other hand, $w_3(w_1, s_2) \leq w_+$, then, by Lemma 4.2,

$$G(w_1, s_2) > G(w_1, \overline{s}_2) > G(\overline{w}_1, \overline{s}_2) = F(\overline{w}_1, \overline{w}_+, \overline{s}_2) = u_+ - u_-. \tag{4.30}$$

(4.30) implies, by virtue of Lemma 4.1, that $\{w_1, w_2, s_2\}$ is not a solution of (1.1), which contradicts our assumption.

Case 2. $w_1 < \overline{w}_1$. For the same reason as for Case 1, we can assume, without loss of generality, that $w_3(w_1, s_2) \leq w_+$. A calculation of

$$\frac{d}{dw}\left(-\frac{p(w)-p(u)}{w-u}\right)$$

shows that $s_2 > \overline{s}_2$ Then, by virtue of Lemma 4.2,

$$G(w_1, s_2) < G(\overline{w}_1, s_2) < G(\overline{w}_1, \overline{s}_2) = F(\overline{w}_1, w_+, \overline{s}_2) = u_+ - u_-. \tag{4.31}$$

Thus, $\{w_1, w_2, s_2\}$ is not a solution of (1.1). We again get a contradiction.

Case 3. $w_1 = \overline{w}_1$. Theorem 3.5 states that, in this case, $w_3(w_1, s_2) > w_+$ and hence $\{w_1, w_2, s_2\}$ is not a solution of (1.1).

Our discussion of the above three cases proves assertion (i).

(ii) The proof of (ii) is similar to that of (i). □

Theorem 4.5. (1.1) *has a unique centered wave solution which is admissible according to the viscosity-capillarity travelling wave criterion.*

Proof. The existence part of our theorem is given by Theorem 2.1.

To prove the uniqueness of the solution of (1.1), it suffices to show that cases (i) and (ii) in Lemma 4.4 are mutually exclusive. Assume the contrary, i.e., there are solutions of (1.1), $\{w_1, w_2, s_2\}$ and $\{\overline{w}_1, \overline{w}_2, \overline{s}_2\}$, with $s_2 \geq 0$ and $\overline{s}_2 < 0$. Then

$$G(w_1, s_2) = J(\overline{w}_2, \overline{s}_2) = u_+ - u_-. \tag{4.32}$$

By Lemma 3.6,

$$w_1 \leq m, \qquad \overline{w}_2 > M, \tag{4.33}$$

where m and M are the Maxwell constants defined by (3.4). Similar to what we did in the proof of Theorem 4.4, we can show that

$$u_+ - u_- = G(w_1, s_2) \leq G(m, 0) = J(M, 0) < J(\overline{w}_2, \overline{s}_2) = u_+ - u_-,$$

which is impossible. □

The uniqueness of the solution $\{w_1, w_2, s_2\}$ of (1.1) enables us to think of w_1, w_2, s_2 as functions of $u_\pm, w_\pm$:

$$w_k = w_k(u_-, u_+, w_-, w_+), \qquad k = 1, 2,$$
$$s_2 = s_2(u_-, u_+, w_-, w_+).$$

For convenience, we shall denote the solution of (1.1) by $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ in the rest of this paper.

Lemma 4.6. (i) *Let* $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ *and* $\{\overline{w}_1, \overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm\}$ *be two solutions of* (1.1a, b) *with* $s_2 \geq 0, \overline{s}_2 \geq 0$. *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if*

$$|u_- - \bar{u}_-| + |u_+ - \overline{u}_+| + |w_- - \overline{w}_-| + |w_+ - \overline{w}_+| < \delta \tag{4.34}$$

then $|w_k - \overline{w}_k| < \varepsilon$, $k = 1, 2$, $|s_2 - \overline{s}_2| < \varepsilon$.

(ii) *Let* $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ *and* $\{\overline{w}_1, \overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm\}$ *be two solutions of* (1.1a, b) *with* $s_2 \leq 0, \overline{s}_2 \leq 0$. *For any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* (4.34) *is satisfied then* $|w_i - \overline{w}_i| < \varepsilon$, $k = 1, 2$, $|s_2 - \overline{s}_2| < \varepsilon$.

Proof. We define an auxiliary function as follows:

$$K(w_1, s_2, u_\pm, w_\pm) := G(w_1, s_2) - u_+ - u_-. \tag{4.35}$$

By Lemma 4.1, a necessary condition for $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ to be a solution of (1.1a, b) is

$$K(w_1, s_2, u_\pm, w_\pm) = 0. \tag{4.36a}$$

We also have

$$K(\overline{w}_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) = 0. \tag{4.36b}$$

Without loss of generality, we assume

$$\overline{w}_1 \le w_1. \tag{4.37}$$

Case 1.

$$\overline{s}_2 \ge s_2. \tag{4.38}$$

Consider the following equation:

$$\begin{aligned} K(w_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) &- K(w_1, \overline{s}_2, u_\pm, w_\pm) \\ &= K(w_1, s_2, u_\pm, w_\pm) - K(w_1, \overline{s}_2, u_\pm, w_\pm) \\ &\quad + K(w_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) - K(\overline{w}_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm). \end{aligned} \tag{4.39}$$

Note that all the variables in (4.39) are in the domain of definition (4.4) of $G(w_1, s_2)$ and hence that of $K(\cdots)$. Since the function $K(\cdots)$ is continuous in $u_\pm$ and $w_\pm$, for any given $\varepsilon' > 0$, there is $\delta_1 > 0$ such that if (4.34) is satisfied with $\delta = \delta_1$ then

$$|K(w_1, \overline{s}_2, u_\pm, w_\pm) - K(w_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm)| < \varepsilon'. \tag{4.40}$$

Then we have, by Remark 4.1, Lemma 4.2, and (4.37), as well as (4.38), that

$$\begin{aligned} K(w_1, s_2, u_\pm, w_\pm) &- K(w_1, \overline{s}_2, u_\pm, w_\pm) \\ &= \frac{\partial G}{\partial s_2^*}(w_1, s_2^*)\bigg|_{s_2^* \in (s_2, \overline{s}_2)} (s_2 - \overline{s}_2) > 0 \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} K(w_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) &- K(\overline{w}_1, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) \\ &= \frac{\partial G}{\partial w_1^*}(w_1^*, \overline{s}_2)\bigg|_{w_1^* \in (\overline{w}_1, w_1)} (w_1 - \overline{w}_1) > 0. \end{aligned} \tag{4.42}$$

Thus, (4.39) and (4.40) imply

$$0 < K(w_1, s_2, u_\pm, w_\pm) - K(w_1, \overline{s}_2, u_\pm, w_\pm) < \varepsilon', \tag{4.43a}$$

$$\left| \frac{\partial G}{\partial w_1^*}(w_1^*, \overline{s}_2)\bigg|_{w_1^* \in (\overline{w}_1, w_1)} (w_1 - \overline{w}_1) \right| < \varepsilon'. \tag{4.43b}$$

Lemma 3.1, Lemma 4.2, and (4.43b) yield

$$|w_1 - \overline{w}_1| < \varepsilon'/\lambda(m). \tag{4.44}$$

We claim that $\overline{s}_2 < s_2 + \varepsilon$ if (4.34) is satisfied for some $\delta > 0$. To this end, we assume the contrary, i.e., $\overline{s}_2 \ge s_2 + \varepsilon$. We rewrite (4.43) as

$$\begin{aligned} \varepsilon' &> K(w_1, s_2, u_\pm, w_\pm) - K(w_1, \overline{s}_2, u_\pm, w_\pm) \\ &= K(w_1, s_2, u_\pm, w_\pm) - K(w_1, s_2 + \varepsilon/2, u_\pm, w_\pm) \\ &\quad + K(w_1, s_2 + \varepsilon/2, u_\pm, w_\pm) - K(w_1, s_2 + \varepsilon, u_\pm, w_\pm) \\ &\quad + K(w_1, s_2 + \varepsilon, u_\pm, w_\pm) - K(w_1, \overline{s}_2, u_\pm, w_\pm). \end{aligned} \tag{4.45}$$

In the same way as we proceeded to get (4.43), we can derive from (4.45) that

$$\begin{aligned}\varepsilon' &> K(w_1, s_2+\varepsilon/2, u_\pm, w_\pm) - K(w_1, s_2+\varepsilon, u_\pm, w_\pm)\\ &= \frac{\partial G}{\partial s_2^*}(w_1, s_2^*)\Big|_{s_2^*\in(s_2+\varepsilon/2, s_2+\varepsilon)} \varepsilon/2.\end{aligned} \tag{4.46}$$

From the definition of $w_2(w_1, s)$, (4.28a), we can see that

$$s_2 \le \lambda(w_2(w_1, s_2)).$$

By the concavity of $p(w)$ in the region $w > \beta$, we have

$$s < \lambda(w_2(w_1, s)) \quad \text{for } s \in [s_2, +\varepsilon/2, s_2+\varepsilon]. \tag{4.47a}$$

Since $\lambda(w_2(w_1, s))$ is continuous in s, there is $\mu(s_2, \varepsilon) > 0$ such that

$$s \le \lambda(w_2(w_1, s)) - \mu(s_2, \varepsilon). \tag{4.47b}$$

By the same reason, we can further choose $\mu(s_2, \varepsilon) > 0$ such that

$$s_3(w_2(w_1, s), w_+) - s \ge \mu(s_2, \varepsilon) > 0 \quad \text{for } s \in [s_2, +\varepsilon/2, s_2+\varepsilon], \tag{4.48}$$

where

$$s_3(w_2, w_+) := -\frac{p(w_2) - p(w_+)}{w_2 - w_+}.$$

With the help of (4.47), (4.48), and (4.4b), we can estimate (4.16) as follows:

$$\begin{aligned}\left|\frac{\partial G}{\partial s_2}(w_1, s_2)\right| &= \left|\frac{w_1 - w_2}{\lambda^2(w_2) - s_2^2}\left[H(w_2 - w_+)\frac{1}{s_3}(s_3 - s_2)(\lambda^2(w_2) - s_2 s_3)\right.\right.\\ &\qquad\left.\left. + H(w_+ - w_2+)(\lambda(w_2) - s_2)^2\right]\right|\\ &> (\beta - \alpha)\left[H(w_2 - w_+)\frac{(s_3 - s_2)}{2\lambda(w_2)} + H(w_+ - w_2+)\frac{\lambda(w_2) - s_2}{2\lambda(w_2)}\right]\\ &> C(w_1, s_2, \varepsilon) > 0.\end{aligned}$$

Recalling (4.46), we obtain

$$\varepsilon' > C(w_1, s_2, \varepsilon)\varepsilon/2. \tag{4.49}$$

However, ε' can be chosen independently of ε. For example, we can choose

$$\varepsilon' < C(w_1, s_2, \varepsilon)\varepsilon/2. \tag{4.50}$$

Then there exists a $\delta_2 > 0$ such that if (4.34) is satisfied with $\delta = \delta_1$, then (4.46) and hence (4.49) hold, which contradicts our choice of ε'. This contradiction proves our claim.

The fact that $|w_2 - \overline{w}_2| < \varepsilon$ if (4.34) holds for some $\delta = \delta_3 > 0$ is a consequence of the fact that w_2 depends on w_1 and s_2 continuously. Now, we choose

$$\varepsilon' \le \min(\varepsilon\lambda(m), C(w_1, s_2, \varepsilon)\varepsilon/2)$$

and then $\delta = \min(\delta_1, \delta_2, \delta_3)$ is the one needed by our assertion.

Case 2. $0 \le \overline{s}_2 < s_2$. In this case, by Theorem 3.5, $w_1 \to w_3(w_1, s_2)$ rather than $w_1 \to w_2(w_1, s_2)$ is a connection. An inspection of the graph of $p(w)$ tells us that

$$\overline{w}_+ \ge w_3(\overline{w}_1, \overline{s}_2) =: \overline{w}_3 > w_3 := w_3(w_1, s_2) = w_+$$

or

$$|w_3 - \overline{w}_3| < |w_+ - \overline{w}_+|. \tag{4.51}$$

From (4.3), we have

$$s_2^2(w_3 - w_1) = -p(w_3) + p(w_1) \tag{4.52a}$$

and

$$\overline{s}_2^2(\overline{w}_3 - \overline{w}_1) = -p(\overline{w}_3) + p(\overline{w}_1). \tag{4.52b}$$

In the following part of our proof, we shall assume, without loss of generality, that $\delta \leq 1$. After subtracting (4.52a) from (4.52b) and some manipulations on the difference, we obtain

$$\begin{aligned}
&(\lambda^2(w_1^*) - \overline{s}_2^2)(w_1 - \overline{w}_1) + (w_3 - w_1)(s_2^2 - \overline{s}_2^2) \\
&\quad = (p'(w_3^*) - \overline{s}_2^2)(w_3 - \overline{w}_3) \\
&\quad \leq \left(\max_{w \in [w_+, \overline{w}_+]} |p'(w)| + s_2^2\right)|w_3 - \overline{w}_3| \\
&\quad \leq \left(\max_{w \in [w_+, w_+ + 1]} |p'(w)| + s_2^2\right)|w_+ - \overline{w}_+| \\
&\quad := C_2(w_+, s_2)|w_+ - \overline{w}_+|,
\end{aligned} \tag{4.53}$$

where $w_1^* \in (\overline{w}_1, w_1)$ and $w_3^* \in (w_3, \overline{w}_3)$. Each term on the left-hand side of (4.53) is nonnegative; hence

$$0 \leq w_1 - \overline{w}_1 \leq \frac{C_2(w_+, s_2)}{\lambda^2(w_1) - s_2^2}|w_+ - \overline{w}_+| \tag{4.54a}$$

and

$$0 \leq s_2 - \overline{s}_2 \leq \frac{C_2(w_+, s_2)}{2s_2(\beta - \alpha)}|w_+ - \overline{w}_+|. \tag{4.54b}$$

Thus, our assertion holds for this case.

(ii) The proof for (ii) is almost similar to that of (i) except that we have to work on the auxiliary function

$$L(w_2, s_2, u_\pm, w_\pm) := J(w_2, s_2) - u_+ - u_- \tag{4.55}$$

instead of $K(w_1, s_2, u_\pm, w_\pm)$. □

Theorem 4.7. *$w_k(u_-, u_+, w_-, w_+)$, $k = 1, 2$, and $s_2(u_-, u_+, w_-, w_+)$ are continuous functions for $w_- \leq \alpha < \beta \leq w_+$.*

Proof. Let $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ and $\{\overline{w}_1, \overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm\}$ be two solutions of (1.1a,b). Our assertion is equivalent to: for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$$|u_- - \bar{u}_-| + |u_+ - \bar{u}_+| + |w_- - \overline{w}_-| + |w_+ - \overline{w}_+| < \delta, \tag{4.56}$$

then $|w_k - \overline{w}_k| < \varepsilon$, $k = 1, 2$, $|s_2 - \overline{s}_2| < \varepsilon$. To this end, since we have Lemma 4.6, it suffices to show our assertion for the case when $s_2 \geq 0$ and $\overline{s}_2 < 0$. By Lemma 4.1, we have

$$K(w_1, s_2, u_\pm, w_\pm) = 0, \tag{4.57a}$$

$$L(\overline{w}_2, \overline{s}_2, \overline{u}_\pm, \overline{w}_\pm) = 0. \tag{4.57b}$$

It follows from above that

$$\begin{aligned} &K(m, 0, u_\pm, w_\pm) - K(m, 0, \bar{u}_\pm, \overline{w}_\pm) \\ &\quad = K(m, 0, u_\pm, w_\pm) - L(M, 0, \bar{u}_\pm, \overline{w}_\pm) \\ &\quad = K(m, 0, u_\pm, w_\pm) - K(w_1, s_2, u_\pm, w_\pm) \\ &\qquad + L(\overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm) - L(M, 0, \bar{u}_\pm, \overline{w}_\pm). \end{aligned} \tag{4.58}$$

It is clear from Lemma 4.2 that the sum of the first two terms on the right-hand side of (4.58) and that of the last two terms are nonnegative. Thus

$$\begin{aligned} 0 &\le K(m, 0, u_\pm, w_\pm) - K(w_1, s_2, u_\pm, w_\pm) \\ &\le |K(m, 0, u_\pm, w_\pm) - K(m, 0, \bar{u}_\pm, \overline{w}_\pm)|, \end{aligned} \tag{4.59a}$$

$$\begin{aligned} 0 &\le L(\overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm) - L(M, 0, \bar{u}_\pm, \overline{w}_\pm) \\ &\le |K(m, 0, u_\pm, w_\pm) - K(m, 0, \bar{u}_\pm, \overline{w}_\pm)|. \end{aligned} \tag{4.59b}$$

Applying Lemma 4.6 to (4.59), we find that for any $\varepsilon > 0$, there is a $\delta > 0$ such that if (4.56) is satisfied then

$$\begin{aligned} |w_1 - m| &< \varepsilon/2, & |w_2 - M| &< \varepsilon/2, \\ |s_2| &< \varepsilon/2, & |\overline{s}_2| &< \varepsilon/2, \\ |\overline{w}_1 - m| &< \varepsilon/2, & |\overline{w}_2 - M| &< \varepsilon/2. \end{aligned}$$

These inequalities imply our theorem. □

Theorem 4.8. *Let* $(u(\xi), w(\xi))$ *be the solution of* (1.1). *For any* $\varepsilon > 0$ *and* $\gamma > 0$, *there is a* $\delta > 0$ *such that if*

$$|u_- - \bar{u}_-| + |u_+ - \bar{u}_+| + |w_- - \overline{w}_-| + |w_+ - \overline{w}_+| < \delta \tag{4.60}$$

then

$$\operatorname{meas}\{\xi \in \mathbb{R} \mid |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \overline{w}(\xi)| \ge \varepsilon\} < \gamma, \tag{4.61}$$

where $(\bar{u}(\xi), \overline{w}(\xi))$ *is the solution of* (1.1a, b) *with Riemann initial values* $(\bar{u}_-, \overline{w}_-)$ *and* $(\bar{u}_+, \overline{w}_+)$.

Proof. Solutions $(u(\xi), w(\xi))$ and $(\bar{u}(\xi), \overline{w}(\xi))$ can be written as $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ and $\{\overline{w}_1, \overline{w}_2, \overline{s}_2, \bar{u}_\pm, \overline{w}_\pm\}$ respectively. We know from Theorem 4.7 that w_1, w_2, s_1, s_2, s_3 are continuous functions of $u_\pm, w_\pm$. Thus there is a $1 \ge \delta_0 > 0$ such that

$$\sum_{k=0}^{3}(|u_k - \bar{u}_k| + |w_k - \overline{w}_k|) + \max_{w \in [w_- - 1, w_+ + 1]} \left| \int_w^{w+1} \lambda(\eta)\, d\eta \right| < \varepsilon \tag{4.62}$$

if (4.60) is satisfied with $\delta = \delta_0$, where $(u_0, w_0) := (u_-, w_-)$, $(u_3, w_3) := (u_+, w_+)$.

We define

$$\begin{aligned} A_\varepsilon := \{\xi \in \mathbb{R} \mid w(\xi) \le \alpha \text{ or } \overline{w}(\xi) \le \alpha, \text{ such that} \\ |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \overline{w}(\xi)| \ge \varepsilon\}, \end{aligned} \tag{4.63a}$$

$$\begin{aligned} B_\varepsilon := \{\xi \in \mathbb{R} \mid w(\xi) \ge \alpha \text{ or } \overline{w}(\xi) \ge \alpha, \text{ such that} \\ |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \overline{w}(\xi)| \ge \varepsilon\}. \end{aligned} \tag{4.63b}$$

The solution $(u(\xi), w(\xi))$ consists of at most two continuous pieces in the region $\{\xi \in \mathbb{R} \mid w(\xi) \leq \alpha\}$. From the results in §2, we can express $\{w_1, w_2, s_2, u_\pm, w_\pm\}$ as follows: If $w_1 < w_-$, then

$$(u(\xi), w(\xi)) = \begin{cases} (u_-, w_-) & \text{for } \xi \in (-\infty, s_1), \\ (u_1, w_1) & \text{for } \xi \in (s_1, s_2). \end{cases} \tag{4.64a}$$

If $w_1 \geq w_-$, then
(4.64b)

$$(u(\xi), w(\xi)) = \begin{cases} (u_-, w_-) & \text{for } \xi \in (-\infty, -\lambda(w_-), \\ (u_1, w_1) & \text{for } \xi \in (-\lambda(w_1), s_2), \\ \multicolumn{2}{l}{\text{for } \xi \in (-\lambda(w_-), -\lambda(w_1)),\ (u(\xi), w(\xi)) \text{ is determined}} \\ \multicolumn{2}{l}{\quad \text{by } \xi^2 = -p'(w(\xi)) \text{ and } u(\xi) = u_- + \int_{w_-}^{w(\xi)} \lambda(\eta)\, d\eta.} \end{cases}$$

In the rest of our proof, we assume that $\delta \leq \delta_0$ and (4.60) holds.

Case 1. $w_1 < w_-$ and $\overline{w}_1 < \overline{w}_-$. In this case,

$$A_\varepsilon \subset (\min(s_1, \overline{s}_1), \max(s_1, \overline{s}_1)) \cup (\min(s_2, \overline{s}_2), \max(s_2, \overline{s}_2))$$

and hence

$$\operatorname{meas} A_\varepsilon \leq |s_1 - \overline{s}_1| + |s_2 - \overline{s}_2|. \tag{4.65}$$

Case 2. $w_1 \geq w_-$ and $\overline{w}_1 \geq \overline{w}_-$. For this case,

$$\begin{aligned} A_\varepsilon \subset\ & (\min(-\lambda(w_-), -\lambda(\overline{w}_-)), \max(-\lambda(w_-), -\lambda(\overline{w}_-))) \\ & \cup (\min(-\lambda(w_1), -\lambda(\overline{w}_1)), \max(-\lambda(w_1), -\lambda(\overline{w}_1))) \\ & \cup (\min(s_2, \overline{s}_2), \max(s_2, \overline{s}_2)) \end{aligned}$$

and thus

$$\operatorname{meas} A_\varepsilon \leq |\lambda(w_-) - \lambda(\overline{w}_-)| + |\lambda(w_1) - \lambda(\overline{w}_1)| + |s_2 - \overline{s}_2|. \tag{4.66}$$

For both cases, since w_1, s_1, w_2 are continuous functions of $u_\pm, w_\pm$, we can find a $\delta_1 > 0$ such that the right-hand sides of (4.65) and (4.66) are less than $\gamma/2$ if (4.60) holds for $\delta = \delta_1$.

Case 3.

$$w_1 < w_-, \qquad \overline{w}_1 \geq \overline{w}_-. \tag{4.67}$$

In this case,

$$\begin{aligned} A_\varepsilon \subset\ & (\min(s_1, -\lambda(\overline{w}_-)), \max(s_1, -\lambda(\overline{w}_-))) \\ & \cup (\min(s_1, -\lambda(\overline{w}_1)), \max(s_1, -\lambda(\overline{w}_1))) \cup (\min(s_2, \overline{s}_2), \max(s_2, \overline{s}_2)) \end{aligned}$$

and thus

$$\begin{aligned} \operatorname{meas} A_\varepsilon & \leq |s_1 - \lambda(\overline{w}_-)| + |s_1 - \lambda(\overline{w}_1)| + |s_2 - \overline{s}_2| \\ & = |\lambda(w^*) - \lambda(\overline{w}_-)| + \lambda(w^*) - \lambda(\overline{w}_1)| + |s_2 - \overline{s}_2|, \end{aligned} \tag{4.68}$$

where $w^* \in (w_1, w_-)$. There exists a $\delta_2 > 0$ such that if (4.60) holds with $\delta = \delta_2$ and if $|w^* - \overline{w}_-| < \delta_2$ and $|w^* - \overline{w}_1| < \delta_2$, then the right-hand side of (4.68) is less than $\gamma/2$. We can further choose $0 < \delta_3 < \delta_2$ such that

$$|w_- - \overline{w}_-| + |w_1 - \overline{w}_1| < \delta_2/2 \tag{4.69}$$

if (4.60) is satisfied with $\delta = \delta_3$. Recalling (4.67), we find that

$$\begin{aligned} 0 &\le w_- - w_1 = w_- - \overline{w}_- + \overline{w}_- - \overline{w}_1 + \overline{w}_1 - w_1 \\ &\le w_- - \overline{w}_- + \overline{w}_1 - w_1 \\ &\le |w_- - \overline{w}_-| + |\overline{w}_1 - w_1| < \delta_2/2. \end{aligned} \tag{4.70a}$$

Similarly, we can prove that

$$0 \le \overline{w}_1 - \overline{w}_- \le |w_- - \overline{w}_-| + |\overline{w}_1 - w_1| < \delta_2/2. \tag{4.70b}$$

It follows immediately from (4.60) and (4.70) that

$$|w_1 - \overline{w}_-| < \delta_2, \qquad |w_- - \overline{w}_1| < \delta_2$$

and hence

$$|w^* - \overline{w}_-| < \delta_2, \qquad |w^* - \overline{w}_1| < \delta_2 \tag{4.71}$$

for $w^* \in (w_1, w_-)$. Therefore, by virtue of (4.68), $\operatorname{meas} A_\varepsilon < \gamma/2$ if (4.60) holds for $\delta = \delta_3$.

Case 4. $w_1 \ge w_-$, $\overline{w}_1 < \overline{w}_-$. We claim that $\operatorname{meas} A_\varepsilon < \gamma/2$ if (4.60) holds for some $\delta = \delta_4 > 0$. The proof is similar to that of Case 3.

Also, we can show that $\operatorname{meas} B_\varepsilon < \gamma/2$ if (4.60) is satisfied with some $\delta = \delta_5$. Now, we take $\delta = \min(\delta_0, \delta_1, \delta_3, \delta_4, \delta_5)$ in (4.60), Then

$$\begin{aligned} &\operatorname{meas}\{\xi \in \mathbb{R} \mid |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \overline{w}(\xi)| > \varepsilon\} \\ &\quad \le \operatorname{meas} A_\varepsilon + \operatorname{meas} B_\varepsilon < \gamma. \quad \square \end{aligned} \tag{4.72}$$

Appendix. The Proof of Lemma 2(i)

Proof. Assume, for comtradiction, that $w(\xi_0-) \in (\alpha, \beta)$ for some $\xi_0 \in \mathbb{R}$. We claim that $w(\xi+) = w(\xi_0-)$ for $\xi \in (\xi_0 - \delta, \xi_0)$ for some $\delta > 0$. Indeed, if otherwise, one of the following two cases will occur:

Case (i). $w(\xi)$ is continuous on $(\xi_0 - \delta, \xi_0)$ for some $\delta > 0$ and there is a sequence $\{\xi_n\} \subset (\xi_0 - \delta, \xi_0)$ such that $\xi_n \to \xi_0-$ as $n \to \infty$ and $w(\xi_n+) \ne w(\xi_0-)$.

Case (ii). There is a sequence of points of discontinuity of $(u(\xi), w(\xi))$ such that $\xi_n \to \xi_0-$ as $n \to \infty$.

Case (ii) cannot occur because $w(\xi_n\pm) \in (\alpha, \beta)$ for large n and the Rankine-Hugoniot conditions at $\xi = \xi_n$ cannot hold.

We claim that Case (i) is also impossible. Indeed, we can integrate (1.4) over (ξ_n, ξ_0), to get

$$\xi_0 \frac{\Delta_n u}{\Delta_n w} = p'(\theta) - \frac{1}{\Delta_n w} \int_{\xi_n+}^{\xi_0-} (u(\zeta) - u(\xi_n+))\, d\zeta, \tag{A1a}$$

$$\frac{\Delta_n u}{\Delta_n w} = -\xi_0 - \frac{1}{\Delta_n w} \int_{\xi_n+}^{\xi_0-} [w(\xi_n+) - w(\zeta)]\, d\zeta, \tag{A1b}$$

where $\Delta_n w := w(\xi_0-) - w(\xi_n+) > 0$, $\Delta_n u := u(\xi_0-) - u(\xi_n+)$ and $\theta \in (w(\xi_n+), w(\xi_0-))$. It follows from Lemma 7.4.1 that

$$\xi_0 \lim_{n\to\infty} \frac{\Delta_n u}{\Delta_n w} = p'(w(\xi_0-)), \qquad \lim_{n\to\infty} \frac{\Delta_n u}{\Delta_n w} = -\xi_0.$$

Then we arrive at the contradiction

$$\left(\lim_{n\to\infty} \frac{\Delta_n u}{\Delta_n w}\right)^2 = -p'(w(\xi_0-)) < 0.$$

It follows that there exists ξ_1 and ξ_2 which are points of discontinuity of $(u(\xi), w(\xi))$ such that $\xi_1 < \xi_0 < \xi_2$ and

$$w(\xi) = w(\xi_0-) \in (\alpha, \beta) \quad \text{for } \xi \in (\xi_1, \xi_2).$$

Therefore, according to the travelling wave criterion, both boundary value problems

$$\text{(A2a)} \quad \frac{d^2\hat{w}}{d\zeta^2} = -2\xi_1 \frac{d\hat{w}(\zeta)}{d\zeta} - \xi_1^2(\hat{w}(\zeta) - w(\xi_1+)) - (p(\hat{w}(\zeta)) - p(w(\xi_1+))),$$

$$\text{(A2b)} \quad \hat{w}(-\infty) = w(\xi_1-), \qquad \hat{w}(+\infty) = w(\xi_1+),$$

$$\text{(A2c)} \quad \hat{w}'(\pm\infty) = 0,$$

and

$$\text{(A3a)} \quad \frac{d^2\hat{w}}{d\zeta^2} = -2\xi_2 \frac{d\hat{w}(\zeta)}{d\zeta} - \xi_2^2(\hat{w}(\zeta) - w(\xi_2-)) - (p(\hat{w}(\zeta)) - p(w(\xi_2-))),$$

$$\text{(A3b)} \quad \hat{w}(-\infty) = w(\xi_2-), \qquad \hat{w}(+\infty) = w(\xi_2+),$$

$$\text{(A3c)} \quad \hat{w}'(\pm\infty) = 0$$

have solutions. A straightforward calculation shows that the eigenvalue for the linearized (near $\hat{w}(\zeta) = w(\xi_1+)$) problem of (A2) is $\lambda = -\xi_1 \pm \sqrt{-p'(w(\xi_1+))}$. It is clear that $w(\xi_1+) = w(\xi_2-)$ is a node of (A2). Thus, in order for (A2) to have a solution, it is necessary that $\xi_1 \geq 0$. On the other hand, however, the same analysis shows that the solvability of (A3) implies $0 \geq \xi_2 > \xi_1 \geq 0$. This contradiction proves our assertion.

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